The Numerical Solution of Certain Dual Integral Equations

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SUMMARY
Dual integral equations with Hankel kernels are reduced to simple integral equations whose unknowns are physically meaningful quantities. Numerical methods for solving these integral equations are established and applied to a problem in electrostatics.

1. Introduction

Dual integral equations arise in many areas of mathematical physics, especially in connection with mixed boundary value problems. Analytical methods for obtaining solutions to dual integral equations have been studied by many authors, and a summary of the important results is given in Sneddon's recent book [4]. Much less attention has been paid to numerical methods for solving these problems. The standard approach to solving dual integral equations is to reduce them to Fredholm integral equations of the second kind (Williams [5]), which can be used to obtain series expansions for the solutions provided certain parameters are small. In some cases (e.g. Cooke [1]) the Fredholm equations are used directly to obtain numerical results. The unknowns involved in these Fredholm equations are certain auxiliary functions and further transformations have to be applied to obtain the physically meaningful quantities.

The present paper takes a different approach by developing integral equations directly in terms of physical quantities and uses numerical techniques to solve the equations. The numerical approach, while giving up a certain amount of generality, avoids a great deal of difficult analytical manipulation and thus makes the approach potentially applicable to a wider class of problems.

The main problem treated in this paper is relatively simple and has been solved by other methods, although our results appear to be more extensive and more accurate than previously published solutions. The approach, however, is quite general and has been used by the author to treat more complicated problems [2]. While the investigation of the general applicability of this method is not completed, it is hoped that it will eventually lead to fairly routine ways of obtaining numerical solutions of dual integral equations.

2. Reduction of Dual Integral Equations

We consider the dual integral equation

$$f(r) = \int_0^\infty A(\xi) h(\xi) J_0(\xi r) d\xi , \tag{1}$$

$$g(r) = \int_0^\infty \xi B(\xi) h(\xi) J_0(\xi r) d\xi , \tag{2}$$

where $h(\xi)$ is unknown, and is to be determined by the conditions that

$$f(r) = F(r), \quad \text{for} \quad 0 \leq r \leq a ,$$
$$g(r) = 0 \quad , \quad \text{for} \quad r > a.$$

One is usually not interested in $h(\xi)$, but in the physical quantities. Specifically, one often wants to know $g(r)$ for $0 \leq r \leq a$.

Inverting (2) by Hankel's inversion theorem,

$$B(\xi)h(\xi) = \int_0^\infty \rho g(\rho) J_0(\xi\rho) d\rho = \int_0^a \rho g(\rho) J_0(\xi\rho) d\rho \, .$$

Substituting into (1) we have

$$\int_0^\infty \int_0^a \frac{A(\xi)}{B(\xi)} \rho g(\rho) J_0(\xi\rho) J_0(\xi r) d\rho \, d\xi = F(r) \, , \qquad 0 \leq r \leq a \, .$$

To simplify we multiply by $r(x^2 - r^2)^{-\frac{1}{2}}$ and integrate from 0 to x. Then

$$\int_0^x \int_0^\infty \int_0^a r(x^2 - r^2)^{-\frac{1}{2}} \frac{A(\xi)}{B(\xi)} \rho g(\rho) J_0(\xi\rho) J_0(\xi r) d\rho \, d\xi \, dr = F_1(x) \, , \tag{3}$$

where

$$F_1(x) = \int_0^x r F(r)(x^2 - r^2)^{-\frac{1}{2}} dr \, .$$

Interchanging orders of integration in (3) and using

$$\int_0^x r J_0(\xi r)(x^2 - r^2)^{-\frac{1}{2}} dr = \frac{\sin \xi x}{\xi} \, ,$$

we get

$$\int_0^a \int_0^\infty \rho g(\rho) J_0(\xi\rho) \frac{A(\xi)}{B(\xi)} \frac{\sin \xi x}{\xi} d\rho \, d\xi = F_1(x) \, .$$

A further simplification can be obtained by differentiating with respect to x, and we get finally

$$\int_0^a \rho K(x, \rho) g(\rho) d\rho = F_1'(x) \, , \tag{4}$$

where

$$K(x, \rho) = \int_0^\infty \frac{A(\xi)}{B(\xi)} J_0(\xi\rho) \cos \xi x \, d\xi \, . \tag{5}$$

   Equations (4) and (5) are the basic equations which will be used to obtain numerical results. It should be remarked that the technique used in obtaining these equations is closely related to the fractional integration methods (Sneddon [4], Noble [3]) usually employed; the main difference of the present approach lies in the establishing of an integral equation in terms of the variable g. The analysis also assumed that the orders of integration can be interchanged—an assumption which needs to be verified in each particular case.

3. Charge on an Electrified Disk

This problem has been solved by many authors using a variety of techniques which are summarized by Sneddon ([4], chapter 3). The dual integral equations for the electrified disk with unit radius, held at a unit potential, are a special case of (1) and (2) with

$$A(\xi) = \xi^{-1} \, ,$$
$$B(\xi) = \xi^{-1} \, ,$$
$$F(r) = 1 \, ,$$
$$a = 1 \, .$$

Then, from (5),

$$K(x, \rho) = \int_0^\infty J_0(\xi\rho) \cos \xi x \, d\xi = 0 \, , \qquad\qquad \rho < x \, ,$$
$$= (\rho^2 - x^2)^{-\frac{1}{2}} \, , \qquad \rho > x \, .$$

Thus (4) becomes

$$\int_x^1 \rho(\rho^2 - x^2)^{-\frac{1}{2}} g(\rho) d\rho = \frac{d}{dx} \int_0^x r(x^2 - r^2)^{-\frac{1}{2}} dr = 1 \, ,$$

which can be inverted directly to give

$$g(\rho) = \frac{2}{\pi}(1-\rho^2)^{-\frac{1}{2}}.$$

The charge density $\sigma(\rho)$ on the disk is

$$\sigma(\rho) = \frac{1}{2\pi}g(\rho) = \frac{1}{\pi^2}(1-\rho^2)^{-\frac{1}{2}}.$$

4. The Circular Plate Condenser

We now consider the case of two coaxial, parallel disks, separated by a distance κ, and charged to equal and opposite potentials $V = \pm 1$. The dual integral equations for this case are ([4], p. 233)

$$\int_0^\infty \xi^{-1}(1-e^{-\kappa\xi})h(\xi)J_0(\xi r)d\xi = 1, \qquad 0 < r < 1, \tag{6}$$

$$\int_0^\infty h(\xi)J_0(\xi r)d\xi = 0, \qquad\qquad r > 1. \tag{7}$$

For this case we have

$$\begin{aligned}
K(x, \rho) &= \int_0^\infty (1-e^{-\kappa\xi})J_0(\xi\rho)\cos\xi x\,d\xi \\
&= -I(\rho, x), \qquad\qquad \rho < x, \\
&= (\rho^2-x^2)^{-\frac{1}{2}} - I(\rho, x), \qquad \rho > x,
\end{aligned}$$

where

$$I(\rho, x) = \frac{1}{\sqrt{2}}[s^2+t^2]^{-\frac{1}{2}}[(s^2+t^2)^{\frac{1}{2}}+s]^{\frac{1}{2}},$$

with

$$s = \kappa^2 + \rho^2 - x^2, \qquad t = 2\kappa x.$$

Thus, (4) becomes

$$\int_x^1 \rho(\rho^2-x^2)^{-\frac{1}{2}}g(\rho)d\rho - \int_0^1 \rho I(\rho, x)g(\rho)d\rho = 1, \text{ for } 0 < x < a. \tag{8}$$

5. Numerical Solution of Equation (8)

The unknown $g(\rho)$ is proportional to the charge density; hence from physical considerations we expect that it will have a singularity of the form $(1-\rho^2)^{-\frac{1}{2}}$ near $\rho = 1$. To facilitate the numerical computations we introduce

$$G(\rho) = (1-\rho^2)^{\frac{1}{2}}g(\rho)$$

and rewrite (8) in terms of $G(\rho)$:

$$\int_x^1 \rho(\rho^2-x^2)^{-\frac{1}{2}}(1-\rho^2)^{-\frac{1}{2}}G(\rho)d\rho - \int_0^1 \rho(1-\rho^2)^{-\frac{1}{2}}I(\rho, x)G(\rho)d\rho = 1. \tag{9}$$

The integrands still have singularities, but numerical methods for integrands of this type are easily established using the technique of product integration [6]. To establish the appropriate procedures we divide the range $[0, 1]$ into $2N$ subintervals, separated by points ρ_i, such that

$$\rho_{2i} - \rho_{2i-1} = \rho_{2i+1} - \rho_{2i} = h_i, \qquad i = 1, 2, \ldots N.$$

In each interval $(\rho_{2i-1}, \rho_{2i+1})$ the function $G(\rho)$ is approximated by a quadratic interpolation

polynomial through the points ρ_{2i-1}, ρ_{2i} and ρ_{2i+1}. Then

$$G(\rho) \simeq G_{2i-1} + \frac{\rho-\rho_{2i-1}}{h_i}[2G_{2i}-1.5G_{2i-1}-0.5G_{2i+1}]$$

$$+ \frac{(\rho-\rho_{2i-1})^2}{2h_i^2}[G_{2i-1}-2G_i+G_{2i+1}],$$

for $\rho_{2i-1} \leqq \rho \leqq \rho_{2i+1}$, where G_j denotes $G(\rho_j)$.

If we use this approximation in (9) and satisfy the resulting equations at $\rho_1, \rho_2, ... \rho_{2N+1}$ we get a linear system of equations

$$\sum_{i=1}^{2N+1} W_{n,i}G_i - \sum_{i=1}^{2N+1} V_{n,i}G_i = 1, \qquad n=1, 2, ... 2N+1, \tag{10}$$

where $W_{n,2i} = 2K_1(i, n) - K_2(i, n)$,

$$W_{n,2i+1} = -0.5K_1(i, n)+0.5K_2(i, n)-1.5K_1(i+1, n)+0.5K_2(i+1, n)+K_0(i+1, n),$$

$$W_{2N+1, 2N+1} = \frac{\pi}{2},$$

$$V_{n,2i} = 2L_1(i, n) - L_2(i, n),$$

$$V_{n,2i+1} = -0.5L_1(i, n)+0.5L_2(i, n)-1.5L_1(i+1, n)+0.5L_2(i+1, n)+L_0(i+1, n),$$

with

$$K_m(i, j) = \frac{1}{h_i^m}\int_{\rho_{2i-1}}^{\rho_{2i+1}} \frac{\rho(\rho-\rho_{2i-1})^m}{(\rho^2-\rho_j^2)^{\frac{1}{2}}(1-\rho^2)^{\frac{1}{2}}}d\rho, \qquad j \neq 2i,$$

$$= \frac{1}{h_i^m}\int_{\rho_j}^{\rho_{2i+1}} \frac{\rho(\rho-\rho_{2i-1})^m}{(\rho^2-\rho_j^2)^{\frac{1}{2}}(1-\rho^2)^{\frac{1}{2}}}d\rho, \qquad j = 2i,$$

for $j=1, 2, ... 2N$, $i=(j+1)/2, ... N$, and $K_m(i, j)=0$ otherwise. L_m is defined similarly:

$$L_m(i, j) = \frac{1}{h_i^m}\int_{\rho_{2i-1}}^{\rho_{2i+1}} \frac{\rho I(\rho, \rho_j)(\rho-\rho_{2i-1})^m}{(1-\rho^2)^{\frac{1}{2}}}d\rho,$$

for $j=1, 2, ..., 2N+1$, $i=1, 2, ..., N$, and $L_m(i, j)=0$ otherwise.

If we can evaluate the moment integrals $K_m(i, j)$ and $L_m(i, j)$ then (10) can be solved by standard methods. For many problems such integrals can be evaluated in closed form, but even if this is inconvenient, as in the present case, it is usually easy to compute them numerically. We used a Romberg quadrature scheme, with some modification near points where the integrand is unbounded, to evaluate the integrals to an accuracy of about 10^{-6}. Once $G(\rho)$ is known, we can get $g(\rho)$ and from it the capacitance C by

$$C = \tfrac{1}{2}\int_0^1 \rho(1-\rho^2)^{-\frac{1}{2}}G(\rho)d\rho.$$

For small κ it was also found convenient to use variable h_i. The choice

$$h_i = \tfrac{1}{2}\left[\sin\frac{\pi i}{2N} - \sin\frac{\pi(i-1)}{2N}\right]$$

gave good results.

6. Numerical Results for the Circular Plate Condenser

To compare with previous results we computed the quantity

$$F = \pi C = \frac{\pi}{2}\int_0^1 \rho(1-\rho^2)^{-\frac{1}{2}}G(\rho)d\rho.$$

The most extensive previous results appear to be those of Cooke [1] (also quoted by Sneddon). Cooke did not carry out any computations for $\kappa < 0.1$ because of numerical difficulties. Our results, summarized in Table 1, extend the range to lower κ. Agreement to at least four significant digits with Cooke's results was found in the overlapping cases.

TABLE 1.

Values of F for the Circular Plate Condenser.

N κ	2	4	6	8	12
0.01	78.29911	80.19972	80.41695	80.43130	80.43853
0.05	17.10715	17.22964	17.23128	17.23214	17.23205
0.10	9.21875	9.23231	9.23310	9.23311	
0.40	3.10250	3.10233	3.10230	3.10229	
0.60	2.39564	2.39545	2.39543		
1.00	1.82087	1.82078	1.82078		

The convergence of the results, while somewhat slow as κ becomes very small, is still sufficiently fast to get good results without excessive computer time. Even in the worst case, $\kappa = 0.01$, our results appear to be correct to at least four significant digits.

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